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On Relativistic Electromagnetic Fluid Flows

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In this paper we have explored certain theorems of theoretical interest in the domain of finitely conducting hydromagnetic systems. These theorems elucidate the local behavior of the congruences of magnetic (electric) field lines. Furthermore, a thermally conducting electromagnetic fluid space-time with vanishing conformal divergence is also investigated.

1. INTRODUCTION

In recent years a considerable amount of interest and attention has been directed towards the study of relativistic magnetohydrodynamics (RMHD) field equations given by Lichnerowicz (1967). Yodzis (1971) and Banerji (1974) have studied the effect of the magnetic field in galactic cosmogony, gravitational collapse, and pulsar theory. Esposito and Glass (1977) have formulated a set of conditions for "restricted steady state" and proved that an infinitely conducting fluid coupled to the frozen-in magnetic field cannot be electrically neutral in this state. Mason (1977) has obtained two relativistic analogs of Ferraro's law of isorotation under two different assumptions. His assumption (Mason, 1977) of a Killing vector collinear to the magnetic field vector is interesting in the sense that the magnetic field lines become "stiff" (Prasad and Sinha, 1979c) when such a Killing vector exists. The vanishing of the expansion of magnetic field lines implies the conservation (Date, 1976) of magnetic field intensity on each individual magnetic field line in the case of perfectly conducting magnetofluids.

Bray (1974a, b; 1975a, b; 1976) has tried to study the local behavior of the congruence of magnetic field lines using Ellis' method (1971). But Ellis' method is physically meaningful for the congruences of timelike curves, and hence would not be applicable to the study of the congruence

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of magnetic field lines. The present author (1978a, b; 1979a, b) has employed Greenberg's theory (1970a) of spacelike congruences to investigate the local behavior of the congruences of magnetic (electric) field lines. On using Greenberg's theory (1970a), we (Prasad, 1978a) have shown that the magnetic fields always remain frozen-in in a perfectly conducting hydromagnetic configuration. In such configuration, the congruences of fluid flow lines and magnetic field lines are always 2-surface forming (Prasad, 1978b). A generalization (Prasad, 1979b) of Ferraro's law of isorotation is obtained, and thereby it is shown that the magnetic field intensity and fluid rotation both increase to a high value when out of a collapsing star a neutron star is born. Relaxing the assumption of infinite conductivity, we have shown (Prasad, 1979a) that the frozen-in property of magnetic fields can still be preserved provided the electric field and fluid vorticity are orthogonal. Thus the success of Greenberg's theory in hydromagnetic systems invites further investigation of the local behavior of magnetic (electric) field lines.

The purpose of this paper is to obtain a wider spectrum of the behavior of magnetic (electric) field lines using Greenberg's theory of spacelike congruences. In particular, we have investigated few theorems of purely theoretical interest in the domain of RMHD inserting the assumption of finite electrical conductivity. These theorems shed light on the local behavior of magnetic (electric) field lines. Finally we have explored a theorem which holds for an electromagnetic fluid space–time in which the divergence of the conformal curvature tensor vanishes.

2. KINEMATICAL PARAMETERS

In this section we mention briefly the kinematical parameters associated with the congruences of timelike and spacelike curves assuming that the signature of the space–time is -2. The kinematical properties of the fluid flow lines are characterized by the usual decomposition for the rate of change of the flow vector (Ehlers, 1961) u^i:

$$u_{i;j}=\sigma_{ij}+\omega_{ij}+\theta\gamma_{ij}+Du_iu_j \tag{2.1}$$

where σ_{ij}, ω_{ij}, and θ denote, respectively, the shear, rotation, and expansion of the congruence of fluid flow lines and D stands for the directional derivative along these lines.

The covariant derivative of the 4-vector n^i tangential to the spacelike congruence is decomposed according to Greenberg (1970) as

$$n_{i;j}=\overset{*}{\sigma}_{ij}+\overset{*}{\omega}_{ij}+\overset{*}{\theta}\,\overset{*}{\gamma}_{ij}-D^*n_in_j+Dn_iu_j-\left(Dn_ku^k\right)u_iu_j$$
$$+\left(D^*n_ku^k\right)u_in_j+n_{k;j}u^ku_i \tag{2.2}$$

where $\overset{*}{\sigma}_{ij}$, $\overset{*}{\omega}_{ij}$, and $\overset{*}{\theta}$ are interpreted as the shear, rotation, and expansion of the congruence of magnetic field lines, respectively. D^* denotes the directional derivative along the magnetic field lines.

3. SOME CONSEQUENCES OF RMHD FIELD EQUATIONS

In this section we begin to study some consequences of RMHD field equations involving the kinematical parameters which are mentioned in Section 2. First, we introduce a local observer who measures the deformation of the magnetic (electric) field lines in the vicinity of any point P of the space–time. Let U^i be the 4-velocity of this observer at the point P and set

$$U^i = u^i - \lambda^i, \qquad U^i n_i = U^i a_i = 0 \tag{3.1}$$

where λ^i is an arbitrary vector orthogonal to the magnetic (electric) field, n^i the unit magnetic field vector, and a^i the unit electric field vector. Furthermore, the vector λ^i is satisfying the condition

$$\lambda^i \lambda_i - 2u^i \lambda_i = 0 \tag{3.2}$$

The arbitrariness of the vector λ^i gives freedom in our choice of an observer. If we suppose that the observer moving with velocity U^i is comoving with the fluid at the point P, then λ^i vanishes at this point. In this case Greenberg's transport law (1970a) for the vector λ^i along the magnetic (electric) field lines is given by

$$-D^*\lambda^k = Dn^k - D^*u^k - (u^i Dn_i)u^k - (u^i D^* n_i)n^k \tag{3.3}$$

and

$$-\hat{D}\lambda^k = Da^k - \hat{D}u^k - (u^i Da_i)u^k - (u^i \hat{D} a_i)a^k \tag{3.4}$$

respectively. $\hat{D}$ denotes directional derivative along the electric field lines.

From the Maxwell field equations, we have recently obtained the following two relations (Prasad, 1979b):

$$\underset{u}{£} D^i + 3\theta D^i - u^i D^j_{;j} + |h| \overset{*}{\alpha}^i - 2u^i h_j \omega^j = J^i \tag{3.5}$$

and

$$\underset{u}{£} B^i + 3\theta B^i - u^i B^j_{;j} - |e| \hat{\alpha}^i - 2u^i e_j \omega^j = 0 \tag{3.6}$$

where D^i is the electric displacement vector, J^i the electric current vector, B^i the magnetic induction vector, e^i the electric field vector, and h^i the magnetic field vector. The vectors $\overset{*}{\alpha}{}^i$ and $\hat{\alpha}^i$ are respectively, defined as

$$\overset{*}{\alpha}{}^i = 2\overset{*}{\omega}{}^i + \overset{*}{\varepsilon}{}^{ij}\{(\ln|h|)_{,j} + D^* n_j - Du_j\} \tag{3.7}$$

and

$$\hat{\alpha}^i = 2\hat{\omega}^i + \hat{\varepsilon}^{ij}\{(\ln|e|)_{,j} + \hat{D}a_j - Du_j\} \tag{3.8}$$

where $\overset{*}{\omega}{}^i$ is the "magnetic vorticity" vector, $\hat{\omega}^i$ the "electric vorticity" vector, $\overset{*}{\varepsilon}{}^{ij}$ the permutation tensor (or an alternating tensor) onto 2-space quotient to the fluid flow and magnetic field lines, and $\hat{\varepsilon}^{ij}$ an alternating tensor onto 2-space quotient to the fluid flow and electric field lines. Now we shall establish the following result.

> *Theorem (3.1).* The "magnetic vorticity" vanishes when the magnetic and electric fields are orthogonal, and the fluid flow and electric field lines are 2-surface forming.

Proof. Splitting (3.5) orthogonal to u^i and D^i and making use of (3.4), we obtain

$$|D|\hat{D}\lambda^k - |h|\hat{\gamma}_i^k \overset{*}{\alpha}{}^i = 0 \tag{3.9}$$

If the fluid flow and electric field lines are 2-surface forming, then $\hat{D}\lambda^k = 0$. In this case (3.9) reduces to

$$\hat{\gamma}_i^k \overset{*}{\alpha}{}^i = 0 \tag{3.10}$$

Contracting (3.10) with n_k and making use of (3.7), we get

$$n_i \overset{*}{\omega}{}^i = 0 \tag{3.11}$$

where $\overset{*}{\omega}{}^i$ is defined by

$$\overset{*}{\omega}{}^i = \tfrac{1}{2}\eta^{ijkl} u_j \overset{*}{\omega}_{kl} \tag{3.12}$$

which yields

$$\overset{*}{\omega}{}^i \eta_{ilkm} = -\left[u_l \overset{*}{\omega}_{km} + u_k \overset{*}{\omega}_{ml} + u_m \overset{*}{\omega}_{lk}\right] \tag{3.13}$$

Operating η^{kmab} on the resulting equation obtained by the contraction of (3.13) with n^l, we have

$$\overset{*}{\omega}{}^a n^b - \overset{*}{\omega}{}^b n^a = 0 \tag{3.14}$$

Contracting (3.14) with n_b and making use of (3.11), we get

$$\overset{*}{\omega}{}^a = 0 \tag{3.15}$$

which proves the statement.

> *Theorem (3.2).* The "electric vorticity" vanishes when the magnetic and electric fields are orthogonal, and the fluid flow and magnetic field lines are 2-surface forming.

The proof of this theorem runs on the lines of the proof of theorem (3.1).

We now wish to introduce the concept of "restricted steady state" (Esposito and Glass, 1977) in case of a finitely conducting magnetofluid. For this purpose, we suppose that the finitely conducting magnetofluid is steadily rotating. In this case the fluid flow vector and the electric field vector may be expressed in terms of a pair of nonorthogonal commuting Killing vectors ξ^i and η^i as follows:

$$u^i = \alpha\xi^i + \beta\eta^i, \qquad e^i = \gamma\xi^i + \delta\eta^i \tag{3.16}$$

where ξ^i and η^i are timelike and spacelike Killing vectors, respectively. Furthermore, the fluid flow vector is Lie transported along the vectors ξ^i and η^i. The scalars γ and δ are defined by (see Glass, 1977)

$$\delta := \xi^i u_i, \quad -\gamma := \eta^i u_i \tag{3.17}$$

which ensures the orthogonality of e^i and u^i. α, β, γ, and δ are explicitly related by a relation

$$\alpha\delta - \beta\gamma = 1 \tag{3.18}$$

The orbits of ξ^i and η^i form a two-dimensional manifold with induced metric tensor $\hat{\gamma}_{ij}$. In a manner of Glass (1977), one may immediately obtain the following results:

$$\theta = 0; \qquad \sigma_{ij} = \tfrac{1}{2}\alpha^2\left(e_i \overset{*}{W}_j + e_j \overset{*}{W}_i\right) \tag{3.19}$$

where $\overset{*}{W}_i$ is the "differential rotation" (Glass, 1977) vector defined by

$$\overset{*}{W}_i = (\beta/\alpha)_{,i}, \qquad e^i \overset{*}{W}_i = 0 \tag{3.20}$$

where β and α are functionally independent since $\sigma_{ij} \neq 0$. We, further, assume that the electric and magnetic fields are orthogonal and choose an orthonormal tetrad vectors, $\lambda^i_{[i]}$ defined (Glass, 1977) by

$$\lambda^i_{[0]} = u^i, \qquad \lambda^i_{[1]} = (2)^{-1/2}(a^i + l^i)$$

$$\lambda^i_{[2]} = (2)^{-1/2}(a^i - l^i) \qquad \text{and} \qquad \lambda^i_{[3]} = n^i \tag{3.21}$$

where l^i is the spacelike unit vector along the "differential rotation arm" (Glass, 1977) which is orthogonal to the unit magnetic field vector. In view of the second relation of (3.19) and (3.21), we observe that the shear tensor of the fluid flow lines has two eigenvectors $\lambda^i_{[1]}$ and $\lambda^i_{[2]}$ with nonzero eigenvalues lying in the 3-space and an eigenvector in the direction of the magnetic field with zero eigenvalue lying in the 2-space. Since the magnetic and electric fields are orthogonal, it is obvious from (3.21) that the magnetic field vector lies in the 2-space quotient to the fluid flow and electric field lines. At this level we remark that the vanishing of shear tensor implies the vanishing of the "differential rotation arm" since the electric field vector cannot vanish due to finite electrical conductivity of the fluid. The magnetic field lies in a nonshearing spatial direction and there is a differential rotation, having a "differential rotation arm" orthogonal to the electric and magnetic fields.

In view of above arguments we define "restricted steady state" as

$$\gamma^i_j DB^j = \gamma^i_j DD^j = \theta = 0 \tag{3.22}$$

which has the following two classes of "restricted steady" motions: (i) a differential rotation with the magnetic field as an eigenvector (orthogonal to the "differential rotation arm") with zero eigenvalue, i.e.,

$$\sigma_{ij} B^j = 0 \tag{3.22$'$}$$

and (ii) rigid rotation, i.e.,

$$\sigma_{ij} = 0 \tag{3.22$''$}$$

which implies the vanishing of the "differential rotation arm." We deduce the following result.

Theorem (3.3). The magnetic field and fluid vorticity are aligned when the fluid flow and magnetic field lines are 2-surface forming, and the electromagnetic fluid with frozen-in magnetic fields orthogonal to electric fields is in "restricted steady state."

Proof. Using (3.22) and (3.22′) in the relation (Prasad, 1979b)

$$\gamma_k^i DB^k - \left(\sigma_k^i + \omega_k^i\right) B^k + 2\theta B^i - |e|\hat{\alpha}^i = 0 \tag{3.23}$$

we get

$$\omega_{ik} B^k + |e|\hat{\alpha}_i = 0 \tag{3.24}$$

Operating η^{ilmn} on both sides of (3.24), we obtain

$$\begin{aligned} B^l(u^m\omega^n - u^n\omega^m) + B^m(u^n\omega^l - u^l\omega^n) \\ + B^n(u^l\omega^m - u^m\omega^l) + 2|e|\left[u^l\hat{\omega}^{mn} + u^m\hat{\omega}^{nl} + u^n\hat{\omega}^{lm}\right] \\ + |e|\left[\beta^l(u^m a^n - u^n a^m) + \beta^m(u^n a^l - u^l a^n)\right. \\ \left. + \beta^n(u^l a^m - u^m a^l)\right] = 0 \end{aligned} \tag{3.25}$$

where

$$\beta^l := (\ln|e|)^{;l} + \hat{D}a^l - Du^l$$

Using the orthogonality condition of magnetic and electric fields in the resulting equation obtained by the contraction of (3.25) with $u_l B_m$, we get

$$|B|^2\omega^n + B^n(\omega^m B_m) + 2|e|\hat{\omega}^{mn}B_m - |e|\,\beta^m B_m a^n = 0 \tag{3.26}$$

Again contracting (3.26) with a_n, we obtain

$$|B|^2\omega^n a_n - |e|\,\beta^m B_m = 0 \tag{3.27}$$

Now the frozen-in property of the magnetic fields implies that the electric field and fluid vorticity are orthogonal (Prasad, 1979b). Thus (3.27) reduces to

$$\beta^m B_m = 0 \qquad \text{as } |e| \neq 0 \tag{3.28}$$

Using (3.28) and the theorem (3.2) in (3.26), we get

$$|B|^2\omega^n + (\omega^m B_m)B^n = 0 \tag{3.29}$$

which proves the statement.

In view of this theorem we conclude that the fluid vorticity must be collinear to the magnetic field and orthogonal to the electric field when the electromagnetic fluid is in "restricted steady state" and "Maxwellian surfaces" (1978b, 1979b) exist.

> *Theorem (3.4).* The expansion of the congruence of electromagnetic energy flux lines vanishes when the electric field is orthogonal to the magnetic field, and the magnetic (electric) field lines are geodesics.

Proof. On using the assumption that the electric field is orthogonal to the magnetic field in (3.14), we obtain

$$\overset{*}{\omega}{}^a e_a = 0 \tag{3.30}$$

Similarly, the counterpart of (3.30) can be written as

$$\hat{\omega}^a h_a = 0 \tag{3.31}$$

Using (3.30) and (3.31) in the divergence identity (Prasad, 1979 b) for the electromagnetic energy flux vector

$$V^i_{;i} = (\ln|V|)_{,i}V^i + 2\left\{|h|e_i\overset{*}{\omega}{}^i - |e|h_i\hat{\omega}^i\right\} + V^i(\hat{D}a_i + D^*n_i - Du_i) \tag{3.32}$$

we get

$$V^i_{;i} + V^i Du_i = (\ln|V|)_{,i}V^i + V^i(\hat{D}a_i + D^*n_i) \tag{3.33}$$

Making use of Greenberg's definition of expansion for the spacelike congruence formed by the electromagnetic energy flux lines in (3.33), we have

$$2|V|\check{\theta} = V^i(\hat{D}a_i + D^*n_i), \qquad \text{as } |V| \neq 0 \tag{3.34}$$

where $\check{\theta}$ is the scalar expansion of the congruence of electromagnetic energy flux lines, $\hat{D}a_i$ the curvature vectors of the electric field lines, D^*n_i the curvature vectors of the magnetic field lines, and $|V|$ the magnitude of the electromagnetic energy flux vector. Equation (3.34) proves the statement made in Theorem (3.4). Here we remark that the last conditions of this theorem can also be replaced by saying that the electromagnetic energy flux vector is orthogonal to the curvature vectors of the electric (magnetic) field lines.

> *Theorem (3.5).* The variation of the total energy density of a viscous and thermally conducting electromagnetic fluid along the flow is balanced by the generation of the energy flux when the fluid flow lines are Born rigid and the acceleration is collinear to the fluid vorticity.

Proof. We have recently obtained (Prasad, 1979b) the equation of continuity for a self-gravitating, viscous, and thermally conducting electromagnetic fluid in the following form:

$$D\overset{*}{\rho}+3\theta\left(\overset{*}{\rho}+\overset{*}{p}\right)+\sigma_{ij}(\lambda e^{i}e^{j}+\mu h^{i}h^{j})$$
$$+2\theta(\lambda|e|^{2}+\mu|h|^{2})-2\nu\sigma^{2}-P^{i}Du_{i}+P^{i}_{;i}=0 \quad (3.35)$$

where

$$\overset{*}{\rho}=\rho+\tfrac{1}{2}(\lambda|e|^{2}+\mu|h|^{2}) \quad (3.36)$$

$$P^{i}=q^{i}-V^{i} \quad (3.37)$$

and

$$q^{i}=K(T_{0,j}-T_{0}Du_{j})\gamma^{ij} \quad (3.38)$$

Here ρ is the matter energy density of the fluid, q^i the heat energy flux vector, $\overset{*}{\rho}$ the total energy density of the electromagnetic fluid, P^i the energy flux vector, $\nu(\geqslant 0)$ the coefficient of viscosity, K the heat conduction coefficient, and T_0 the rest temperature. The relation (Prasad, 1979b) between the energy flux vector P^i and the kinematical parameters associated with the fluid flow lines is given by

$$P^{i}=\gamma^{i}_{j}\left(\omega^{jk}_{;k}-\sigma^{jk}_{;k}+2\theta^{,j}\right)+\omega^{i}_{k}Du^{k}+\sigma^{i}_{k}Du^{k} \quad (3.39)$$

Contracting (3.39) with Du_i and making use of our assumptions, i.e.,

$$\sigma_{ij}=\theta=\omega_{ik}Du^{k}=0 \quad (3.40)$$

we obtain

$$P^i Du_i = -(Du_j)_{;k}\omega^{jk} \tag{3.41}$$

Again using (3.40) together with the identities (Oliver and Davis, 1977; Greenberg, 1970b)

$$\underset{u}{\pounds}\,\omega_{ij} = 0 \tag{3.42}$$

$$D\omega^2 + 4\theta\omega^2 + 2\sigma_{ij}\omega^i\omega^j = \omega^{ij}\underset{u}{\pounds}\,\omega_{ij} \tag{3.43}$$

and

$$D\omega^2 = -4\theta\omega^2 - 2\sigma_{ij}\omega^i\omega^j + (Du_i)_{;j}\omega^{ij} \tag{3.44}$$

in (3.41), we get

$$P^i Du_i = 0 = D\omega^2 \tag{3.45}$$

Making use of first two conditions of (3.40) in (3.35) and combining the resulting equation with (3.45), we obtain

$$D\overset{*}{\rho} = -P^i_{;i} \tag{3.46}$$

which proves the statement. The last relation of (3.45) shows that the magnitude of the fluid vorticity vector is conserved on the congruence of fluid flow lines.

An alternating tensor $\hat{\varepsilon}^{ij}$ is defined by

$$\hat{\varepsilon}^{ij} = \eta^{ijkl}u_k a_l \tag{3.47}$$

By virtue of (2.1), (2.2) for the vector a^i and (3.47), we get

$$\hat{\varepsilon}^{ij}_{;j} = 2\hat{\omega}^i - 2u^i a_k\omega^k - \hat{\varepsilon}^{ik}(Du_k - \hat{D}a_k) \tag{3.48}$$

where

$$\hat{\omega}^i = \tfrac{1}{2}\eta^{ijkl}u_j\omega_{kl} \tag{3.49}$$

From (3.48), we obtain the divergence identity for the "electric vorticity" vector,

$$\hat{\omega}^i_{;i} + \hat{\omega}^i Du_i - \hat{\omega}^i Da_i - (a_k\omega^k u^i)_{;i} + u^k\hat{D}a_k a_i\omega^i + \tfrac{1}{2}\hat{\varepsilon}^{ki}\{(Du_k)_{;i} - (\hat{D}a_k)_{;i}\} = 0 \tag{3.50}$$

For a physical interpretation of (3.50), we assume that the electromagnetic fluid space–time admits two Killing vector fields, one of them collinear to the fluid flow vector and the other proportional to the unit electric field vector. The existence of a Killing vector collinear to the fluid flow vector implies (Banerji, 1974) that

$$\theta = \sigma_{ij} = 0, \qquad Du_i = -(\ln \xi)_{,i} \tag{3.51}$$

where ξ is the magnitude of the Killing vector ξ^i representing an analog of the Newtonian potential. Since η_i is a Killing vector collinear to the unit electric field vector a^i, one can write

$$\eta_i = \varphi a_i, \qquad \eta_{i;j} + \eta_{j;i} = 0 \tag{3.52}$$

where φ is any nonzero scalar. Let us write

$$\hat{\gamma}_i^l \hat{\gamma}_j^k (\eta_{l;k} + \eta_{k;l}) = \varphi \hat{\gamma}_i^l \hat{\gamma}_j^k (a_{l;k} + a_{k;l}) \tag{3.53}$$

Using (3.52) in the left-hand side of (3.53), we get

$$\hat{\gamma}_i^l \hat{\gamma}_j^k a_{(l;k)} = 0 \tag{3.54}$$

which may equivalently be written as

$$\hat{\sigma}_{ij} + \hat{\theta} \hat{\gamma}_{ij} = 0 \tag{3.55}$$

which shows that the congruence of electric field lines is "stiff" (Prasad and Sinha, 1979c) because the shear and expansion of the congruence of electric field lines vanish identically when a spacelike Killing vector collinear to the electric field is admitted. Again (3.52) yields

$$\varphi_{,i} a_j + \varphi_{,j} a_i + \varphi(a_{i;j} + a_{j;i}) = 0 \tag{3.56}$$

Contracting (3.56) with a^i, we get

$$(\hat{D}\varphi) a_j - \varphi_{,j} + \varphi D a_j = 0 \tag{3.57}$$

Again contracting (3.57) with a^j, we obtain

$$\hat{D}\varphi = 0 \tag{3.58}$$

Using (3.58) in (3.57), we have

$$\hat{D} a_j = (\ln \varphi)_{;j} \tag{3.59}$$

By means of (3.50), the second relation of (3.51), (3.59), and the frozen-in property (Prasad, 1979 b) of the magnetic fields, we obtain

$$\hat{\omega}^i_{;i} + \hat{\omega}^i Du_i - \hat{\omega}^i \hat{D} a_i = 0 \tag{3.60}$$

Using Greenberg's definition of the expansion for the spacelike congruence formed by "electric vortex" lines in (3.60), we get

$$\underset{v}{D}\left(\ln \hat{\omega} \underset{v}{A}\right) = m^i \hat{D} a_i \tag{3.61}$$

where m^i is the unit "electric vorticity" vector, $\hat{\omega}$ the magnitude of the "electric vorticity" vector, $\underset{v}{D}$ the directional derivative along the "electric vortex" lines, and $\underset{v}{A}$ the proper area subtended by the "electric vortex" lines as they pass through the screen in the 2-space quotient to the fluid flow and the "electric vortex" lines. From (3.61), we conclude that the "electric vorticity" flux through any loop moving with the finitely conducting fluid is constant and that the fluid particles which lie initially on an "electric vortex" line continue to do so when the electric field lines are geodesics or when the "electric vorticity" vector is orthogonal to the curvature vector of the electric field line provided our assumptions hold. This may also be called as the frozen-in property of the "electric vorticity" in the sense of Alfven (1950).

4. *C* SPACE CONTAINING THERMALLY CONDUCTING ELECTROMAGNETIC FLUID

A space–time in which the divergence of conformal curvature tensor vanishes is called *C* space in the sense of Szekeres (1964). The property of *C* space is characterized by an identity (Szekeres, 1964)

$$R_{i[j;k]} - \tfrac{1}{6} g_{i[j} R_{,k]} = 0 \tag{4.1}$$

where R_{ij} is the usual Ricci tensor and R the scalar curvature of the space–time. Square brackets denote skew symmetrization.

We now assume that the space–time is filled with thermally conducting electromagnetic fluid. The stress-energy–momentum tensor (Prasad, 1979 b) for thermally conducting electromagnetic fluid is given by

$$T_{ij} = \left(\rho^* + p^*\right) u_i u_j - p^* g_{ij} - (\lambda e_i e_j + \mu h_i h_j) + P_i u_j + P_j u_i \tag{4.2}$$

It follows from (4.2) and well-known Einstein's field equations that

$$R_{ij} = -Au_iu_j + Bg_{ij} + (\lambda e_ie_j + \mu h_ih_j) - P_iu_j - P_ju_i \tag{4.3}$$

where

$$A = \rho^* + p^* \qquad \text{and} \qquad 2B = (\rho - p + \lambda|e|^2 + \mu|h|^2)$$

We establish the following theorem inserting the assumption that the electric and magnetic fields are orthogonal.

> *Theorem (4.1).* If a thermally conducting electromagnetic fluid space–time with harmonic energy flux vector is a C space, then one of the following holds: (i) projection of fluid rotation tensor vanishes in 2-space quotient to the fluid flow and electric field lines when the fluid vorticity and magnetic field are aligned, (ii) projection of fluid rotation tensor vanishes in 2-space quotient to the fluid flow and magnetic field lines when the fluid vorticity and electric field are aligned.

Proof. Since a thermally conducting electromagnetic fluid space–time is a C space, it follows from (4.1) and (4.3) that

$$u_iA_{,[j}u_{k]} + Au_{i;[j}u_{k]} + Au_iu_{[k;j]} + g_{i[j}B_{,k]} - \lambda e_{i;[j}e_{k]} + \lambda e_ie_{[j;k]}$$
$$- \mu h_{i;[j}h_{k]} + \mu h_ih_{[j;k]} + P_{i;[j}u_{k]} + P_iu_{[k;j]} - u_iP_{[j;k]} + u_{i;[j}P_{k]} = 0 \tag{4.4}$$

Contracting (4.4) with u^i and making use of the assumption that the energy flux vector P_i is a harmonic vector, i.e., $P_{[i;j]} = 0$, we obtain

$$A_{,[j}u_{k]} + Au_{[k;j]} + u_{[j}B_{,k]} - \lambda u^ie_{i;[j}e_{k]} - \mu u^ih_{i;[j}h_{k]} + u^iP_{i;[j}u_{k]} = 0 \tag{4.5}$$

Splitting up (4.5) orthogonal to u^i, we get

$$A\omega_{lm} - \lambda u^ie_{i;[l}e_{m]} - \lambda(e_{i;k}u^iu^k)u_{[m}e_{l]} - \mu u^ih_{i;[l}h_{m]} - \mu(h_{i;k}u^iu^k)u_{[m}h_{l]}$$
$$- u^iP_{i;[l}u_{m]} - u^iP_{i;[l}u_{m]} - u_{[l}P_{m];k}u^k = 0 \tag{4.6}$$

Again splitting up (4.6) orthogonal to u^i and n^i both, we obtain

$$A\,\overset{*}{\gamma}{}^l_i\,\overset{*}{\gamma}{}^m_j\,\omega_{lm} - \lambda\,\overset{*}{\gamma}{}^l_i\,\overset{*}{\gamma}{}^m_j\,u^ke_{k;[l}e_{m]} = 0 \tag{4.7}$$

Projecting (4.7) orthogonal to u^i and a^i both, we finally get

$$\hat{\gamma}_l^i \hat{\gamma}_m^j \omega_{ij} + \tfrac{1}{2} n^i \omega_{i[l} n_{m]} = 0 \tag{4.8}$$

which reduces to

$$\hat{\gamma}_l^i \hat{\gamma}_m^j \omega_{ij} = 0 \tag{4.9}$$

when the fluid vorticity and magnetic field are aligned, i.e., $n^i \omega_{il} = 0$. Hence (4.9) proves the first statement. Similarly, the counterpart of (4.8) may be obtained as

$$\overset{*}{\gamma}{}_l^i \overset{*}{\gamma}{}_m^j \omega_{ij} + \tfrac{1}{2} a^i \omega_{i[l} a_{m]} = 0 \tag{4.10}$$

which reduces to

$$\overset{*}{\gamma}{}_l^i \overset{*}{\gamma}{}_m^j \omega_{ij} = 0 \tag{4.11}$$

when the electric field and fluid vorticity are aligned. This proves the second statement. This theorem gives a type of physical information that follows when it is known that a given space–time containing thermally conducting electromagnetic fluid is a C space. However, we have in no way attempted systematically to examine all consequences of C space in this case. Further results can be investigated.

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